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A SIMPLE PROCEDURE FOR CONSTRUCTING SOLUTIONS OF NONLINEAR HEAT-CONDUCTION PROBLEMS BY THE KANTOROVICH METHOD

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A simplified procedure based on expansion in the neighborhood of an approximate solution is discussed for solution of the quasilinear heat-conduction equation.

It is generally known that either the energy method or the more promising method of Galerkin [2] is used in connection with the method of Kantorovich [1]. The crux of either approach is that in the solution of nonlinear problems of mathematical physics one must inevitably cope with systems of nonlinear ordinary differential equations and algebraic equations, a prospect that often incurs insurmountable difficulties and naturally imposes limitations on practical applications. A vital problem in this connection is the search for a procedure that can be used to construct solutions of nonlinear problems by reduction to ordinary differential equations without having to solve systems of nonlinear equations, at least in the stage of refinement of the initial approximation.

Below we consider such a procedure for the quasilinear heat-conduction equation in three-dimensional space and for a general type of nonlinear boundary-value problem. In addition to the requirements of existence and uniqueness of a solution, we impose constraints that are quite strong, but are nonetheless frequently justified, as a rule, in a number of practical problems, as for example in the area of heat physics: 1) The solution T (x, y, z, t) is representable with sufficient practical accuracy in some neighborhood of a certain initial approximation $T = T_0(x, y, z, t)$ by an equation in the form of a power series, finite or infinite, which is differentiable with respect to the coordinates and time; 2) in the neighborhood of $T = T_0(x, y, z, t)$ the coefficients in the equation and in the boundary conditions are analytical functions of T.

Consider the equation

$$f_1(T)\frac{\partial T}{\partial t} = \nabla [f_2(T)\nabla T] \quad (1)$$

subject to the boundary conditions on the surface s

$$f_3(T)\nabla T + f_4(T)|_s = 0. \quad (2)$$

In accordance with constraints 1) and 2) we represent T and the functions f_i (i = 1, 2, 3, 4) in the form

$$T = \alpha_0 + \varepsilon\alpha_1 + \ldots \varepsilon^n\alpha_n, \quad (3)$$

$$f_i = f_i|_{T=T_0} + \left.\frac{\partial f_i}{\partial T}\right|_{T=T_0}(T - T_0) + \ldots = \beta_{ki}(\varepsilon^m\alpha_m)^k. \quad (4)$$

The "Einstein rule," i.e., summation with respect to a certain index, is tacitly understood at all times.

As the initial expression for α_0 we can take the solution given by, for example, the integral [3, 4] or any other suitable method.

We substitute (3) and (4) into (1) and (2). We obtain

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$$\beta_{k1}(\varepsilon^m\alpha_m)^k\varepsilon^p\frac{\partial\alpha_p}{\partial t}=\nabla\{\beta_{k2}(\varepsilon^m\alpha_m)^k\varepsilon^p\nabla\alpha_p\};\qquad(5)$$

$$\beta_{k3}(\varepsilon^m\alpha_m)^k\varepsilon^p\nabla\alpha_p+\beta_{k4}(\varepsilon^m\alpha_m)^k|_s=0;\qquad(6)$$

$$p,\ k,\ n=0,\ 1,\ 2,\ \ldots,\ \infty,$$

$$m=1,\ 2,\ \ldots,\infty.$$

Equating coefficients of like powers of ε on the right- and left-hand sides of (5) and setting them equal to zero in (6), we obtain the following array of equations and corresponding boundary conditions:

$$\beta_{01}\frac{\partial\alpha_1}{\partial t}+\beta_{11}\alpha_1\frac{\partial\alpha_0}{\partial t}=\nabla\{\beta_{02}\nabla\alpha_1+\beta_{12}\alpha_1\nabla\alpha_0\};\qquad(7)$$

$$\beta_{03}\nabla\alpha_1+\beta_{14}\alpha_1\nabla\alpha_0+\beta_{14}\alpha_1|_s=0;\qquad(8)$$

$$\beta_{01}\frac{\partial\alpha_2}{\partial t}+\beta_{11}\alpha_2\frac{\partial\alpha_0}{\partial t}+\beta_{11}\alpha_1\frac{\partial\alpha_1}{\partial t}+\beta_{21}\alpha_1^2\frac{\partial\alpha_0}{\partial t}=\nabla\{\beta_{02}\nabla\alpha_2+\beta_{12}\alpha_2\nabla\alpha_0+\beta_{12}\alpha_1\nabla\alpha_1+\beta_{22}\alpha_1^2\nabla\alpha_0\};\qquad(9)$$

$$\beta_{03}\nabla\alpha_2+\beta_{13}\nabla\alpha_0+\beta_{13}\alpha_1\nabla\alpha_1+\beta_{23}\alpha_1^2\nabla\alpha_0+\beta_{14}\alpha_2+\beta_{24}\alpha_1^2|_s=0,\qquad(10)$$

. . . .

Equations (7) and (9) and their counterparts for other α_i with the boundary conditions (8) and (10) are linear and, hence, much simpler to solve than the original nonlinear problem (in particular, by the same technique of reduction to ordinary differential equations). We are also aware [1] that the convergence of the variational method of Kantorovich for linear systems is determined by the already proved [1, 2] convergence of the Ritz method. Thus, if the series (3) turns out to be convergent, the solution constructed by the foregoing scheme will converge to the exact solution. A similar statement applies to the case in which the method of Galerkin is adopted as the direct method. It is sufficient for practical purposes, in light of the rapid convergence of the corresponding series, to retain only the first (one or two) terms of the Kantorovich representations. The same is true of the series (3).

As an illustration we consider the problem [5]

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left[(1+\gamma T)\frac{\partial T}{\partial x}\right];\qquad(11)$$

$$T(0,\ t)=1;\quad T(x,\ 0)=0.$$

For small variations the system (11) can be used, for example, as a simplified version for analysis of the behavior of the baseline in the nonlinear theory of differential hot-wire anemometry for a constant specific heat and linear temperature dependence of the thermal conductivity.

We limit the series (3) to the first three terms. We adopt as α_0 the solution obtained by the integral method:

$$\alpha_0=1-\frac{2x}{\delta}+\frac{x^2}{\delta^2};\quad\delta=\sqrt{12(1-\gamma)t}.\qquad(12)$$

We seek α_1 and α_2 in accordance with the Kantorovich method [1] as a series with respect to the relatively complete system of functions χ_{ki}:

$$\alpha_i=\sum_{k=1}^{\infty}\chi_{ki}(x,\ t)\,a_{ki}(t);$$

$$\chi_{ki}=(\delta-x)\,x^k$$

$$i=1,\ 2,$$

which satisfy zero-valued boundary conditions; we stop with the first approximation, i.e.,

$$\alpha_1=(\delta-x)\,x\alpha_1,\qquad(13)$$

$$\alpha_2=(\delta-x)\,xa_2.\qquad(14)$$

We substitute (12)-(14) into (7) and (8), which we then insert into the functional

$$J(\alpha_i)=(L_i\alpha_i,\ \alpha_i)-(f_i,\ \alpha_i)-(\alpha_i,\ f_i);$$

$$L\alpha_1=\frac{\partial\alpha_1}{\partial t}-\frac{\partial}{\partial x}\left\{(1+\gamma\alpha_0)\frac{\partial\alpha_1}{\partial x}+\gamma\alpha_1\frac{\partial\alpha_0}{\partial x}\right\};$$

$$L\alpha_2 = \frac{\partial \alpha_2}{\partial t} - \frac{\partial}{\partial x}\left\{(1+\gamma\alpha_0)\frac{\partial \alpha_2}{\partial x} + \gamma\alpha_2\frac{\partial \alpha_0}{\partial x}\right\}; \tag{15}$$

$$f_1 = 0;\ f_2 = \frac{\partial}{\partial x}\left\{\gamma\alpha_1\frac{\partial \alpha_1}{\partial x}\right\}.$$

Forming the Euler equations from (15) and using zero-valued initial conditions, we obtain

$$\alpha_1 = \frac{x(\delta - x)}{4}\, t^{\frac{35-9\gamma}{24(\gamma-1)}};$$

$$\alpha_2 = \frac{3(1-\gamma)}{4(2+5\gamma)}\,[1 + t^{\frac{23-3\gamma}{12(\gamma-1)}}]\, x(\delta - x).$$

As a result, the approximate solution of the nonlinear problem (11) has the form

$$T = 1 - \frac{2x}{\sqrt{12(1-\gamma)t}} + \frac{x^2}{12(1-\gamma)t} + \varepsilon\frac{x(\delta - x)}{4}\left\{t^{-\frac{35-9\gamma}{24(1-\gamma)}} + \varepsilon\frac{3\gamma(1-\gamma)}{2+5\gamma}(1 + t^{-\frac{23-3\gamma}{12(1-\gamma)}}\right\}. \tag{16}$$

The values of T calculated according to (16) for, say, $\gamma = 0.1$ and various values of x and t differ from the corresponding exact solutions [5] only in the third place.

We note that the same result can be obtained more simply if the Galerkin method is taken as the direct one.

NOTATION

x, y, z, coordinates; t, time; ε, small parameter; α, coefficients in expansion of solution T into power series in ε; f_i, coefficients in starting equation and boundary conditions; β, coefficients in expansion of f into series in ε; χ_{ki}, basis functions; a_{ki}, unknown functions in the Kantorovich representation; γ, constant coefficient.

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